Chapter 1 : Construction of Solutions for Two Dimensional Riemann Problems

From the construction, it is Solutions for two-dimensional Riemann problems clear that the above shock obeys the conditions of Theorem (b) and hence obeys the entropy condition.

This paper investigates the structure of two-dimensional Riemann problems for Hamiltonâ€"Jacobi equations. The solutions to such problems are fundamental building blocks for constructing solutions to more general problems, in particular, for numerical construction using methods such as front tracking. Here we prove the existence of a particular class of Riemann prob- lem for which the viscosity solutions contain closed characteristic orbits, enclosing furthermore a periodic sonic structure, which in turn encloses a parabolic structure. This investigation was prompted by the discovery of numerical evidence of examples displaying an even richer internal structure. Here DS denotes the gradient of S: Assuming only continuity of H, 1 , 2 has a unique viscosity solution S see [9] based on [4, 5, 14] ; nevertheless the construction of the solution is rarely straightforward. We develop a geometric framework for understanding the structure of solutions and singularities arising therein. Within this framework we exhibit a class of problems for which the viscosity solution has a complex internal structure. The work here establishes tools for the construction of a variety of classes of solution types. Equation 3 occurs in many contexts as an evolution model, e. A natural framework for understanding the evolution of singularities in solutions to 3 is to seek them in self-similar form: The theory of Legendre transforms has its parallel in the construction of solutions to 3 , i. These ideas have been further extended to certain nonconvex cases [3]. Such results allow for the decomposition of an arbitrary function H as a sum of simpler ones and have led to Godunov-type algorithms for the numerical construction of viscosity solutions for 3 [1, 19, 21]. Such algorithms yield then, in principle, numerical algorithms for the construction of solutions to the Riemann problem 1 , 2. The front tracking [7, 8, 10, 12] approach to 3 is opposite in spirit, as it uses knowledge of Riemann solutions to propagate singularities in the solution and higher order methods for the propagation of the solution where it is smooth. Although the local structure of Riemann solutions for Hamiltonâ€"Jacobi equations is well understood [9], a complete theory of their global structure and algorithms for their construction is lacking at the time of this writing. It is hoped that the example studied here, by providing insight to the structure of solutions, will further this goal, as in the case of Riemann problems for the related two-dimensional conservation laws [17, 26, 28]. Within a compact set, piecewise characteristic closed paths can form, along which the Cauchy problem fails to be hyperbolic. A shock is called sonic if characteristics are tangent to it. Sonic shocks can be characteristic, and thus straight lines, or noncharacteristic, and thus curved. The main result of this paper is the following theorem. There exist C 1 Hamiltonians and Riemann problems so that the associated Riemann solutions to the Hamiltonâ€"Jacobi equation 1 possess a period 4 sonic sequence. Conjectured structure of viscosity solution to 4 , 5. Ci , Ri , P are domains of plane waves, sonic rarefactions, and a parabolic wave. Regions Ci are the domains of plane waves see below , on which the solution is linear; regions Ri are the domains of sonic rarefactions, on which the solution is a union of lines tangent to the curves Vi ; and the region P is the domain of a parabolic wave, on which the solution is a saddle-type function. To prove Theorem 1 we construct a somewhat simpler function S containing a period 4 sonic sequence, see Figure 2, and from it derive H and data DS such that S is a Riemann solution. Characteristics form two closed paths, one joining points Bi , the other joining points Ai , and the shock segments between these points form the sonic sequence. We remark that period 3 sonic sequences are trivial since a closed path of three characteristics lies on a straight line. In section 2 we provide relevant background material. In section 3 we give an ex- ample for which the Riemann solution contains a closed characteristic path bounding a parabolic wave, later used in section 7. Section 4 describes the geometric structure of sonic rarefactions and in section 5 we derive properties of sonic sequences required for the construction of periodic sonic sequences in section 6. The proof of Theorem 1 is found in section 7, where we simultaneously construct S, H, and Riemann prob- lems. The author would like to thank Tangerman for suggesting this problem and his insights toward its resolution. Solution with a period 4 sonic sequence, used in the proof of Theorem 1. The solution here is simpler than that in Figure 1; there, characteristics leave the parabolic wave. For any given Riemann data and continuous H, 1 , 2 has a unique Lipschitz con- tinuous Riemann solution S [4, 9, 14]. Outside a compact set these equations are nonsingular and the Riemann data can be propagated inward by hyperbolic methods [9]. They are directed toward points given by the parabolic mapping L p: A characteristic path can terminate at a shock or, if the shock is sonic, continue along the outgoing tangential shock. By an orbit we mean a closed path of characteristics. As a consequence, characteristics satisfy the Lax condition [16] at a shock [9]. Characteristic directions point toward the shock or are tangent to it. Let E be an open connected set on which a Riemann solution is C 2. Throughout the paper we exploit the symmetry of 1. Example with period 4 orbit. This example furnishes us with a structure from which we can construct outward a solution containing a sonic sequence, as done in the following sections, and which is ultimately used to prove Theorem 1. The idea is to replace the plane waves with sonic rarefactions tangential to the plane waves at the parabolic boundary. The boundary of the parabolic wave is an orbit of period 4. These shocks are also the boundaries of the plane waves and hence S is continuous. Riemann solution with orbit. Ci , P are domains of plane and parabolic waves, respectively. Structure of sonic rarefactions. Sonic rarefactions occur when character- istics leave a shock V tangentially. Since characteristics are straight lines, a sonic rarefaction is a type of ruled surface, called the tangent surface of the curve V. The properties developed here, and in section 5 sequences of sonic rarefactions , allow for the construction in section 6 of a period 4 sonic sequence, using an orbit as starting point as from, e. Before proceeding, we provide basic terminology. For a convex curve v: The interior of the intersection of all such half planes of v is called the inside of v. The outside of v is the closure of the complement of the inside of v. Note that the inside of a convex planar curve is convex. We now describe the tangent surface to a curve; see Figure 4. Since our con- struction of the periodic sequence in section 6 is outward from an orbit, the choice of orientation in 8 is the natural one. For convex v, S is a function representing the tangent surface. The inside of v is the intersection of half planes containing v. We show in Proposition 2 that for some H, S is a solution to 1. Of principal importance in the following lemma is the orthogonality of v and p, leading naturally to a notion of duality, and further exploited in developing properties of sonic sequences, and in the construction of our solution. Evalu- ating this we obtain 9 , and since this is independent of s, the gradient is constant along characteristics. This result allows us to construct our solution in terms of pairs v, p: Then v, p is called an orthogonal pair. If v and p are also convex, then v, p is called a dual pair. Let S be a tangent surface with dual pair v, p. Characteristics tangent to P or p are dual to those tangent to V or v. Let H be dual to S. A periodic structure on the left, with annular domain on the right. Sequences of sonic rarefactions. If V1 is a curve embedded in the tangent surface of a curve V0 , we can form a sequence of two tangent surfaces by truncating characteristics from V0 along V1. Here we derive properties of sequences of length 2, 3, and 4, necessary for the construction carried out in section 6. A periodic structure satisfying 1 for some H is a period 4 sonic sequence. How- ever, a period 4 sonic sequence need not be a periodic structure: The numerical solution [23, 22] motivating Theorem 1 contains such a sequence, joining a periodic structure to a parabolic wave. In a period 3 orbit the three characteristics must lie in a plane, hence the gradient values on each characteristic lie along a straight line. Thus period 3 sonic structures are trivial. The periodic sonic sequence constructed in section 6 is bounded by orbital paths and in the following proposition we establish the geometry of such paths in the v- and p-planes. Let S be a periodic structure. Note that these equations are independent of curves V1 and V3 i. These dual pairs allow for the construction of curves V0 and V2 , and associated tangent surfaces, unique up to additive constants. From these, the curves V1 and V3 hence the periodic structure are derived. If the denominators are nonzero, the system 13 , 14 is globally Lipschitz in pi t and by the methods of Chapters 25 and 26 of [6] and Chapter 2 of [11] has a unique solution on I which is a C 2 function of initial conditions, parameters, and t. The dual pairs generate a dual periodic structure O representing chords of super- and subderivatives of S. Construction of dual pairs. Let v0 , v2: The following lemma is implied by assumptions

A1 and A2. Integrating from 0 to t gives A similar argument establishes This is precisely the geometry needed to construct curves V1 and V3. Before proceeding with the construction we need to show that the curves p0 and p2 are regular. Q is the sector generated by q. The curves p0 and p2 resulting from Proposition 4 are regular.

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Chapter 2 : CiteSeerX â€” Citation Query Construction of solutions for two-dimensional Riemann problems

In 52, the two dimensional nonlinear waves are described for the class of problems f, fz. In 53 a method for explicitly constructing the 2-dimensional solutions in this class from the nonlinear waves is described. Complete solutions for 2-dimensional Riemann problems for representative forms of the.

Wave Fronts for Hamilton-Jacobi Equations: Tangerman , " The Hamilton-Jacobi equation describes the dynamics of a hypersurface in R n. This equation is a nonlinear conservation law and thus has discontinuous solutions. The dependent variable is a surface gradient and the discontinuity is a surface cusp. Here we investigate the intersection of cusp hypersu Here we investigate the intersection of cusp hypersurfaces. We propose the class of Hamilton-Jacobi equations as a natural higher-dimensional generalization of scalar equations which allow a satisfactory theory of higher-dimensional Riemann problems. The first main result of this paper is a general framework for the study of higherdimensional Riemann problems for Hamilton-Jacobi equations. The purpose of the framework is to understand the structure of Hamilton-Jacobi wave interactions in an explicit and constructive manner. Specialized to two-dimensional Riemann problems i. We are specifically interested in models of deposition and etching, important processes for Show Context Citation Context Riemann solutions are commonly used for systems in one space dimension. We are further interested in those Riemann solutions which form an elementary wave pattern, generalizing the notion of shock waves and center In this thesis we examine overturned solutions of scalar partial differential equations in one and two dimensions using moving nite element methods with particular emphasis on scalar conservation laws. These equations are the simplest nonlinear equations to exhibit the formation of shocks and expans These equations are the simplest nonlinear equations to exhibit the formation of shocks and expansions as their solutions evolve with time. Both analytic and numerical techniques are examined in one and two dimensions, analytic techniques being considered as a background to the numerical methods, which are adaptive and finite element in nature. They include the classical moving finite element method MFE of Miller in its various forms and Lagrangian methods. The analytic and numerical solution to these equations yield multivalued curves and surfaces. Weak solutions however exist in which shocks feature and these can be obtained from the multivalued solutions by applying a recovery technique to locate the shock position. Pinezich , " This paper investigates the structure of two-dimensional Riemann problems for Hamilton-Jacobi equations. We show that it is possible for the viscosity solution to contain closed characteristic orbits, enclosing furthermore a periodic sonic structure, which in turn encloses a parabolic structure. The existence of such examples elucidates the difficulties encountered in designing construction methods for viscosity solutions to Riemann problems in dimension 2. This investigation was prompted by the discovery of numerical evidence of examples displaying an even richer internal structure. Here we establishe the existence of Riemann problems with viscosity solutions of considerable complexity. Show Context Citation Context Solution with a period 4 sonic sequence.